

On group choosability of total graphs

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Abstract

In this paper, we study the group and list group colorings of total graphs and we give two group versions of the total and list total colorings conjectures. We establish the group version of the total coloring conjecture for the following classes of graphs: graphs with small maximum degree, two-degenerate graphs, planar graphs with maximum degree at least 11, planar graphs without certain small cycles, outerplanar and near-outerplanar graphs. In addition, the group version of the list total coloring conjecture is established for forests, outerplanar graphs and graphs with maximum degree at most two.

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1 Introduction

Throughout this paper, we consider simple graphs, the graphs without loops and multiple edges and we follow [7] for terminology and notation not defined here. For a graph G , we denote its vertex set, edge set, maximum degree and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V , E , Δ , δ), respectively. If $v \in V(G)$, we use $\deg_G(v)$ (or simply $\deg(v)$) and $N_G(v)$ to denote the degree and neighborhoods of v in G , respectively.

A *proper coloring* of G is a coloring of the vertices of G so that no two adjacent vertices are assigned the same color. The minimum number of colors in any proper

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coloring of G , $\chi(G)$, is called the *chromatic number* of G . A k -list assignment for a graph G is a function L which assigns to each vertex v of G a list of colors $L(v)$ such that $|L(v)| = k$. An L -coloring of G is a proper coloring c such that $c(v) \in L(v)$ for each vertex v . If for every k -list assignment L , a proper L -coloring of G exists, then G is said to be k -choosable and the *choice number*, $\chi_l(G)$, of G is the smallest k such that G is k -choosable. The concepts of *chromatic index*, $\chi'(G)$, and *choice index*, $\chi'_l(G)$, can be defined similarly in terms of coloring the edges of G .

Recall that the *total graph* of a graph G , denoted by $T(G)$, is a graph where its vertices are the edges and vertices of G and adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G . The *total chromatic number* of G , $\chi''(G)$, is the chromatic number of $T(G)$. Clearly $\chi''(G) = \chi(T(G)) \geq \Delta(G) + 1$. Behzad [1] and Vizing [15] posed independently the following famous conjecture, which is known as the *total coloring conjecture*.

Conjecture 1 For any graph G , $\chi''(G) \leq \Delta(G) + 2$.

The *total choice number* of G , $\chi''_l(G)$, is the choice number of $T(G)$. It follows directly from the definition that $\chi''_l(G) \geq \chi''(G)$. The notation of total choosability was first introduced by Borodin et al. [2]. They proposed the following conjecture, known as the *list total coloring conjecture*.

Conjecture 2 For any graph G , $\chi''_l(G) = \chi''(G)$.

The concept of group coloring of graphs was first introduced by Jaeger et al. [10]. Assume that A is a group and $F(G, A)$ denotes the set of all functions $f : E(G) \rightarrow A$. Consider an arbitrary orientation of G . Graph G is called A -colorable if for every $f \in F(G, A)$, there is a vertex coloring $c : V(G) \rightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for each directed edge from x to y . The *group chromatic number* of G , $\chi_g(G)$, is the minimum k such that G is A -colorable for any group A of order at least k . In [6], the concept of group choosability is introduced as an extension of list coloring and group coloring. Let A be a group of order at least k and $L : V(G) \rightarrow 2^A$ be a list assignment of G . For $f \in F(G, A)$, an (A, L, f) -coloring of G is an L -coloring $c : V(G) \rightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for each directed edge from x to y . If for each $f \in F(G, A)$ there exists an (A, L, f) -coloring for G , then we say that G is (A, L) -colorable. If for any group A of order at least k and any k -list assignment $L : V(G) \rightarrow 2^A$, G is (A, L) -colorable, then we say that G is k -group choosable. The *group choice number* of G , $\chi_{gl}(G)$, is the smallest k such that G is k -group choosable. Clearly group choosability of a graph is independent of the orientation on G . The concept of group choosability is also studied in [5]. The authors used the concept of D -group choosability to establish the group version of the Brook's Theorem. A graph G is called D -group choosable if it is (A, L) -colorable for each group A with $|A| \geq \Delta(G)$ and every list assignment $L : V(G) \rightarrow 2^A$ with $|L(v)| = \deg(v)$. They proved the following theorem, which is a characterization of D -group choosable graphs.

Theorem 1.1 ([5]) The connected graph G is D -group choosable if and only if G has a block which is neither a complete graph nor a cycle.

The following result is the group version of the Brook's Theorem.

Theorem 1.2 ([5]) For every connected simple graph G , $\chi_{gl}(G) \leq \Delta(G) + 1$, with equality if and only if G is either a cycle or a complete graph.

We extend the concepts of total and list total colorings to total group and list total group colorings of graphs. We let $\chi_g''(G) = \chi_g(T(G))$ (resp. $\chi_{gl}''(G) = \chi_{gl}(T(G))$) and we called it the *total group chromatic number* (resp. *total group choice number*) of G . Clearly the following inequality holds for the mentioned chromatic numbers of G .

$$\chi_{gl}''(G) \geq \max\{\chi_g''(G), \chi_l''(G)\} \geq \chi''(G).$$

Now we extend the total coloring and list total coloring conjectures as follows:

Conjecture 3 For every graph G , $\chi_{gl}''(G) \leq \Delta(G) + 2$.

Conjecture 4 For every graph G , $\chi_{gl}''(G) = \chi_g''(G)$.

The following conjecture express the weaker version of Conjecture 3.

Conjecture 5 For every graph G , $\chi_g''(G) \leq \chi''(G) + 1$.

In this paper, we are interested in Conjectures 3 and 4 and we will establish conjecture 3 for certain classes of graphs such as planar graphs with maximum degree at least 11, two-degenerate graphs, planar graphs without certain cycles, outerplanar and near-outerplanar graphs. Also we show that Conjecture 4 holds for graphs with maximum degree at most two, forests and outerplanar graphs. Subsequently, it will be shown that Conjecture 5 holds for the above mentioned classes of graphs.

2 Some upper bounds

In this section, we give some upper bounds for $\chi_{gl}''(G)$ of a graph G and use these bounds to verify Conjectures 3 and 4 for some classes of graphs. The *group choice index* of a graph G , $\chi'_{gl}(G)$, is defined as the group choice number of its line graph i.e. $\chi'_{gl}(G) = \chi_{gl}(L(G))$. Clearly $\chi'_{gl}(G) \geq \chi'(G) \geq \Delta(G)$. This concept is studied in [11] where the authors conjectured that every graph with maximum degree Δ is $(\Delta + 1)$ -edge group choosable. Moreover, they gave infinite families of graphs G with $\chi'_{gl}(G) = \Delta(G)$. The following lemma shows that Conjecture 3 holds for these graphs.

Lemma 2.1 For every graph G we have $\chi_{gl}''(G) \leq \chi'_{gl}(G) + 2$.

Proof. Let G be a graph and A be a group of order at least $\chi'_{gl}(G) + 2$. Also let $L : V \cup E \rightarrow 2^A$ be any $(\chi'_{gl}(G) + 2)$ -list assignment of $V \cup E$ and $f \in F(T(G), A)$ be arbitrary. First we color the vertices of G from their lists. Since $\chi'_{gl}(G) \geq \Delta(G)$ and $\chi_{gl}(G) \leq \Delta(G) + 1$ by Theorem 1.2, such a coloring exists. Let $c : V(G) \rightarrow A$ be such a coloring so that $c(v) \in L(v)$. For each edge $e = uv$ of G , without loss of generality, let the edge eu be directed from e to u and also the edge ev be directed from e to v in $T(G)$. For each edge $e = uv$ of G , remove $f(eu) + c(u)$ and $f(ev) + c(v)$ from $L(e)$. Since for each edge e of G , $|L(e)| \geq \chi'_{gl}(G) + 2$, each edge of G retains at least $\chi'_{gl}(G)$ admissible colors in its list and so, by the definition of $\chi'_{gl}(G)$, it is possible to color the edges of G from their lists. So we can color the vertices of $T(G)$ from their lists and this yields an (A, L, f) -coloring of $T(G)$, which shows that $\chi''_{gl}(G) \leq \chi'_{gl}(G) + 2$. ■

The *coloring number* of G , $\text{col}(G)$, is the smallest integer for which there exists an ordering of the vertices of G such that each vertex v has at most $\text{col}(G) - 1$ neighbors among vertices of smaller indices. A graph G is called *d-degenerate* if $d \geq \text{col}(G) - 1$. For a non-regular graph G with maximum degree Δ , it is easy to see that $\text{col}(T(G)) \leq \Delta + \text{col}(G) - 1$. If G is a Δ -regular graph, then $T(G)$ is 2Δ -regular which is not complete or a cycle since $|V(G)| \geq 4$. So by Theorem 1.2, we have $\chi''_{gl}(G) \leq 2\Delta = \Delta + \text{col}(G) - 1$. Hence we have the following result.

Theorem 2.2 Let G be a graph with maximum degree Δ . Then

$$\chi''_{gl}(G) \leq \Delta + \text{col}(G) - 1.$$

Our first application of Theorem 2.2 is the following, which states that Conjectures 3 and 4 hold for all forests.

Corollary 2.3 Let G be a forest with maximum degree $\Delta \geq 2$. Then

$$\chi''_{gl}(G) = \chi''_g(G) = \Delta + 1.$$

A graph H is a *minor* of a graph K if H can be obtained from a subgraph of K by contracting some edges. A graph G is called *K_4 -minor free* if it has no subgraph isomorphic to a minor of K_4 . It is well-known that [8] every K_4 -minor free graph has a vertex of degree at most two. A planar graph is called *outerplanar* if it has a drawing in which each vertex lies on the boundary of the outer face. It is well-known that a graph is outerplanar if and only if it contains neither K_4 nor $K_{2,3}$ as minors (see for example [7]).

It is well known that [14] a connected graph G is 2-choosable if and only if G obtained by successively removing vertices of degree 1 until what remains, is isomorphic to either K_1 , C_{2m+2} or $\Theta_{2,2,2m}$ for some m , where $\Theta_{2,2,2m}$ is the graph consisting of two distinguished vertices v_0, v_{2m} connected by three paths P_1 , P_2 and P_3 of lengths

2, 2 and $2m$, respectively. So the class of two-degenerate graphs properly contain, 2-choosable graphs, outerplanar graphs, non-regular subcubic graphs, planar graphs of girth at least six and *unicycle graphs*, graphs with exactly one cycle, as subclasses. Using Theorem 2.2, we have the following corollary.

Corollary 2.4 Conjecture 3 holds for every two-degenerate graph. In particular, Conjecture 3 holds for planar graphs of girth at least six, K_4 -minor free, outerplanar and 2-choosable graphs.

3 Graphs with bounded degrees

In this section, we prove that Conjectures 3 and 4 hold for graphs with maximum degree at most two and wheel graphs with maximum degree at least five. Also we prove that Conjecture 3 holds for any planar graph with maximum degree at least 11.

Theorem 3.1 Let P_n and C_n be path and cycle on $n \geq 2$ vertices, respectively. Then $\chi''_g(P_n) = \chi''_{gl}(P_n) = 3$ and $\chi''_g(C_n) = \chi''_{gl}(C_n) = 4$.

Proof. Clearly $\chi''_g(P_n) \geq \chi''(P_n) = 3$. On the other hand, $\chi''_{gl}(P_n) \leq \text{col}(T(P_n)) = 3$. It follows that $\chi''_g(P_n) = \chi''_{gl}(P_n) = 3$. Denote by v_1, v_2, \dots, v_n the vertices of C_n in the order they appear in C_n with $u_i = v_i v_{i+1}$ and $T = T(C_n)$. By Theorem 1.2, we have $\chi''_{gl}(C_n) \leq 4$, for any n . Also it is easy to see [16] that $\chi''(C_n) = 3$ if $n = 3t$ and $\chi''(C_n) = 4$, otherwise. Consequently, if n is not a multiple of three, we obtain the desired result. So let $n = 3t$ and $A = (Z_3, +)$ be the group with elements 0,1,2 where "+" is the addition modulo 3. Define $f \in F(T, Z_3)$ with $f(u_{n-1}u_n) = 1$, $f(u_n u_1) = 2$ and $f(e) = 0$ otherwise, where the edge $u_{n-1}u_n$ in T is directed from u_{n-1} to u_n and the edge $u_n u_1$ is directed from u_n to u_1 . Suppose that $c : V(T) \rightarrow Z_3$ is an (Z_3, f) -coloring of the vertices of $T(C_n)$ with $c(v_1) = i$, $c(v_2) = j$ and $c(u_1) = k$, where $0 \leq i \neq j \neq k \leq 2$. Since $f(u_1 u_2) = f(v_2 u_2) = 0$, $c(u_2)$ must be different from $c(u_1) = k$ and $c(v_2) = j$ and hence $c(u_2) = i$. By the same reasoning, $c(v_3) = k$ and since $n = 3t$ we have $c(v_{n-1}) = j$, $c(v_n) = k$ and finally $c(u_{n-1}) = i$. As a consequence, for any choices of i, j, k , since $f(u_{n-1}u_n) = 1$ and $f(u_n u_1) = 2$, the vertex u_n can not admit any admissible color and hence $\chi''_g(C_n) \geq 4$. Therefore for any n , we obtain $\chi''_g(C_n) = 4$. Now the inequality $\chi''_g(C_n) \leq \chi''_{gl}(C_n) \leq 4$, implies that $\chi''_g(C_n) = \chi''_{gl}(C_n) = 4$ which completes the proof. ■

Note that if G is a graph with components G_1, G_2, \dots, G_t , then we have:

$$\chi''_g(G) = \max\{\chi''_g(G_1), \dots, \chi''_g(G_t)\}, \quad \chi''_{gl}(G) = \max\{\chi''_{gl}(G_1), \dots, \chi''_{gl}(G_t)\}.$$

Combining these facts and Theorem 3.1, we obtain the following corollary.

Corollary 3.2 Let G be a graph with maximum degree at most two. Then we have $\chi''_g(G) = \chi''_{gl}(G)$.

The *wheel graph*, W_n , is the graph obtained from C_n by adjoining a vertex to all vertices of C_n . For $n \geq 6$, it is easy to see that $\text{col}(T(W_n)) = n + 1$ and hence $\chi_g''(G) = \chi_{gl}''(G) = n + 1$. So these graphs satisfy Conjectures 3 and 4. It seems that for every graph G with unique vertex of maximum degree, $\chi_g''(G) = \chi_{gl}''(G) = \Delta + 1$.

Theorem 3.3 ([13]) For every planar graph G with minimum degree at least 3 there is an edge $e = uv$ with $\deg(u) + \deg(v) \leq 13$.

Theorem 3.4 Let $k \geq 11$ and G be a planar graph with maximum degree at most k . Then $\text{col}(T(G)) \leq k + 2$.

Proof. Let G be a minimal counterexample for Theorem 3.4. Then there exists a $k \geq 11$ so that $\text{col}(T(G)) > k + 2$. If G contains a vertex u with $\deg(u) \leq 2$, then by the minimality of G we have $\text{col}(T(G - u)) \leq k + 2$. Since the degree of each edge incident to u in $T(G) - u$ is at most $k + 1$ we have $\text{col}(T(G)) \leq k + 2$, which contradicts the choice of G as a counterexample. So we may assume that $\delta \geq 3$. By Theorem 3.3, there exists an edge $e = uv$ with $\deg_{T(G)}(e) \leq 13$. We may assume that $\deg(u) \leq 6$. By minimality of G we have $\text{col}(T(G - e)) \leq k + 2$. Since $\max\{\deg_{T(G)-u}(e), \deg_{T(G)}(u)\} \leq 12$ and $k \geq 11$ we obtain that $\text{col}(T(G)) \leq k + 2$, which is a contradiction. ■

Using Theorem 3.4, we obtain the following corollary, which states that Conjecture 3 holds for planar graphs with maximum degree at least 11.

Corollary 3.5 Let G be a planar graph with maximum degree Δ . Then

$$\chi_{gl}''(G) \leq \text{col}(T(G)) \leq \max\{13, \Delta + 2\}.$$

4 Outerplanar and near outerplanar graphs

In this section, we give some upper bounds for the total group choice number of outerplanar and near outerplanar graphs, which establish Conjectures 3 and 4 for these graphs. We need the following lemma of Borodin and Woodall [3].

Lemma 4.1 Let G be an outerplanar graph. Then at least one of the following holds.

- (a) $\delta(G) = 1$.
- (b) There exists an edge uv such that $\deg(u) = \deg(v) = 2$.
- (c) There exists a 3-face uxy such that $\deg(u) = 2$ and $\deg(x) = 3$.
- (d) There exist two 3-faces xu_1v_1 and xu_2v_2 such that $\deg(u_1) = \deg(u_2) = 2$ and $\deg(x) = 4$ and these five vertices are all distinct.

The following theorem implies that Conjectures 3 and 4 hold for outerplanar graphs with maximum degree at least 5.

Theorem 4.2 Let $k \geq 5$ and G be an outerplanar graph with maximum degree $\Delta \leq k$. Then $\text{col}(T(G)) \leq k + 1$.

Proof. Let G be a minimal counterexample for the theorem. So $\text{col}(T(G)) > k + 1$ for some $k \geq 5$. If G contains a vertex v of degree one with neighborhood u , then by minimality of G we have $\text{col}(T(G - v)) \leq k + 1$. Since $\deg_{T(G)-v}(uv) \leq k$ and $\deg_{T(G)}(v) \leq 2$ we have $\text{col}(T(G)) \leq k + 1$, a contradiction. So by Lemma 4.1, G contains an edges uv such that $\deg(u) = 2$ and $\deg(u) + \deg(v) \leq 6$. By the minimality of G , $\text{col}(T(G - uv)) \leq k + 1$. Again since $k \geq 5$, $\deg_{T(G)-u}(uv) \leq 5$ and $\deg_{T(G)}(u) = 4$, we obtain that $\text{col}(T(G)) \leq k + 1$, a contradiction. ■

Combining Theorems 4.2 and 2.2, we have the following corollary.

Corollary 4.3 Let G be an outerplanar graph with maximum degree $\Delta \neq 4$, then $\chi''_{gl}(G) \leq \max\{5, \Delta + 1\}$. In particular if $\Delta \geq 5$ then $\chi''_{gl}(G) = \chi''_g(G) = \Delta + 1$.

By a *near outerplanar* graph we mean one that is either K_4 -minor free or $K_{2,3}$ -minor free. Near outerplanar graphs are an extension of outerplanar graphs. Theorem 4.5, will show that Conjecture 3 holds for the class of $K_{2,3}$ -minor free graphs. In fact, in Theorem 4.5 we will replace the class of $K_{2,3}$ -minor free graphs by the slightly larger class of $\bar{K}_2 + (K_1 \cup K_2)$ -minor free graphs, where $\bar{K}_2 + (K_1 \cup K_2)$ is the graph obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2. Before we proceed, we need the following lemma which characterizes $\bar{K}_2 + (K_1 \cup K_2)$ -minor free graphs.

Lemma 4.4 ([9]) Let G be a $\bar{K}_2 + (K_1 \cup K_2)$ -minor free graph. Then each block of G is either K_4 -minor free or isomorphic to K_4 .

Theorem 4.5 Let $k \geq 4$ and $G \neq K_4$ be a $\bar{K}_2 + (K_1 \cup K_2)$ -minor free graph with maximum degree $\Delta \leq k$. Then $\text{col}(T(G)) \leq k + 2$.

Proof. Let G be a minimal counterexample for Theorem 4.5 and also let $k \geq 4$ be such that $\text{col}(T(G)) > k + 2$. We may assume that G is connected. First, let G be 2-connected. By Lemma 4.4 we may assume that G is a K_4 -minor free graph. So by Corollary 3.2, G is non-regular and hence $\text{col}(T(G)) \leq \Delta + \text{col}(G) - 1 \leq k + 2$, a contradiction. Hence G is not a 2-connected graph. Suppose B is an end-block of G with cut-vertex v . First let $B \cong K_4$ with $V(B) = \{v, u, w, x\}$. By minimality of G we have $\text{col}(T(G - (B - v))) \leq k + 2$. Since $\deg_H(e) \leq k$ for $e \in N_{T(G)}(v) \cap E(B)$ and $H = T(G) - \{u, w, x, uw, ux, wx\}$, we have $\text{col}(T(G)) \leq k + 2$, a contradiction. So B is K_4 -minor free, and this means that B contains at least two vertices of degree at most 2. Let v be a vertex of degree 2 in B such that $\deg_G(v) = 2$. By minimality of G we have $\text{col}(T(G - v)) \leq k + 2$. Since $\deg_{T(G)-v}(e) \leq k + 1$ for $e \in N_{T(G)}(v)$, we have $\text{col}(T(G)) \leq k + 2$, a contradiction. ■

By Theorem 1.1 we have $\chi''_{gl}(K_4) \leq 6$ and so by Theorem 4.5, Conjecture 3 holds for $K_{2,3}$ -minor free graphs with maximum degree at least 4.

Corollary 4.6 Let G be a $K_{2,3}$ -minor free graph with maximum degree at least 4. Then G is $(\Delta + 2)$ -total group choosable.

Using Corollaries 2.4 and 4.6 we conclude that Conjecture 3 holds for near-outerplanar graphs.

5 Planar graphs without small cycles

In this section, we prove that Conjecture 3 holds for some planar graphs without certain small cycles. Our proofs are based on some structure lemmas and discharging method.

Lemma 5.1 Let G be a graph with $\delta \leq 2$ and $\Delta \geq 3$. If for any $e \in E(G)$, $\text{col}(T(G - e)) \leq \Delta + 2$, then $\text{col}(T(G)) \leq \Delta + 2$.

Proof. Let v be a vertex of degree at most two and e be an edge incident with v . By the hypothesis, $\text{col}(T(G - e)) \leq \Delta + 2$. Since $\deg_{T(G) - v}(e) \leq \Delta + 1$ and $\deg_{T(G)}(v) \leq 4$, we have $\text{col}(T(G)) \leq \Delta + 2$. ■

A *plane graph* is a particular drawing of a planar graph on the plane. A *2-alternating cycle* in a graph G is a cycle of even length in which alternate vertices have degree 2 in G .

Theorem 5.2 ([12]) Let G be a connected planar graph with $\delta \geq 2$. If G contains no 5-cycles nor 6-cycles, then G contains a 2-alternating cycle or an edge uv such that $\deg(u) + \deg(v) \leq 9$.

The following theorem establishes Conjecture 3 for a planar graph with maximum degree at least 7 that contains no 5-cycles or 6-cycles.

Theorem 5.3 Let $k \geq 7$ and G be a planar graph with maximum degree $\Delta \leq k$. If G contains no 5-cycles or 6-cycles, then $\text{col}(T(G)) \leq k + 2$.

Proof. Let $G = (V, E)$ be a minimal counterexample for Theorem 5.3. So for some $k \geq 7$, $\text{col}(T(G)) > k + 2$. If G contains a vertex v of degree one, then by the minimality of G , $\text{col}(T(G - v)) \leq k + 2$. Since $\deg_{T(G) - v}(e) \leq k$ and $\deg_{T(G)}(v) \leq 2$ where e is the edge incident with v , we have $\text{col}(T(G)) \leq k + 2$, a contradiction. So we may assume that $\delta \geq 2$. First suppose that G has an edge $e = uv$ with $\deg(u) + \deg(v) \leq$

9. Without loss of generality, we assume that $\deg(u) \leq 4$. Then $\text{col}(T(G - e)) \leq k + 2$. Since $\deg_{T(G)-u}(e) \leq 8$ and $\deg_{T(G)}(u) \leq 8$, we have $\text{col}(T(G)) \leq k + 2$, a contradiction. Hence for every edge $e = uv$ of G , $\deg(u) + \deg(v) \geq 10$. By Theorem 5.2, G must contain a 2-alternating cycle C . Let U be the set of the vertices of C that have degree 2 in G , and let $H = G - U$. By the minimality of G and as $|V(H)| < |V(G)|$, we have $\text{col}(T(H)) \leq k + 2$. For each $e \in E(C)$ and $v \in U$ of C in G we have $\deg_{T(G)-U}(e) \leq k + 1$ and $\deg_{T(G)}(v) \leq 4$ and so $\text{col}(T(G)) \leq k + 2$, a contradiction. This contradiction completes the proof of the theorem. ■

A cycle C of length k in a graph G is called a k -net if C has at least one chord in G . The following, is a structural lemma for plane graphs without 5-nets.

Lemma 5.4 ([4]) Let G be a planar graph with $\delta \geq 3$ and without 5-nets. Then G contains an edge xy such that $\deg(x) + \deg(y) \leq 9$.

The following theorem establishes Conjecture 3 for every planar graph without 5-nets and maximum degree at least 7.

Theorem 5.5 Let $k \geq 7$ and G be a planar graph with maximum degree $\Delta \leq k$. If G contains no 5-nets, then $\text{col}(T(G)) \leq k + 2$.

Proof. Let $G = (V, E)$ be a minimal counterexample for Theorem 5.5. Then there is a $k \geq 7$ so that $\text{col}(T(G)) > k + 2$. By Lemma 5.1, we may assume that $\delta \geq 3$. Using Lemma 5.4, G contains an edge $e = xy$ such that $\deg(x) + \deg(y) \leq 9$ and $\deg(x) \leq 4$. By the minimality of G we have $\text{col}(T(G - e)) \leq k + 2$. Since $\deg_{T(G)-x}(e) \leq 8$ and $\deg_{T(G)}(x) \leq 8$, we have $\text{col}(T(G)) \leq k + 2$, a contradiction. ■

We denote the set of faces of a plane graph G by $F(G)$ or simply by F . For a plane graph G and $f \in F(G)$, we write $f = u_1 u_2 \dots u_n$ if u_1, u_2, \dots, u_n are the vertices on the boundary walk of f enumerated clockwise. Let $\delta(f)$ denote the minimum degree of vertices incident with f . The *degree of a face* f , denoted by $\deg(f)$, is the number of edge steps in the boundary walk. A k -vertex (resp. k^+ -vertex) is a vertex of degree k (resp. a vertex of degree at least k). The following theorem establishes Conjecture 3 for planar graphs without 4-cycles and maximum degree at least 6.

Theorem 5.6 Let $k \geq 6$ and G be a planar graph with maximum degree $\Delta \leq k$ such that G has no cycle of length 4. Then $\chi''_{gl}(G) \leq k + 2$.

Proof. Let G be a minimal counterexample. So for a $k \geq 6$, a group A with $|A| \geq k + 2$, a $(k + 2)$ -list assignment $L : V(T(G)) \rightarrow 2^A$ and $f \in F(T(G), A)$, $T(G)$ is not (A, L, f) -colorable. Graph G has the following properties:

- (1) G is connected,
- (2) Any vertex v is incident with at most $\lfloor \frac{\deg(v)}{2} \rfloor$ 3-faces,
- (3) The minimum degree of G is at least three that is $\delta \geq 3$,
- (4) G contains no edge uv with $\min\{\deg(u), \deg(v)\} \leq \frac{k}{2}$ and $\deg(u) + \deg(v) \leq k + 2$,
- (5) G does not contain any 3-face $F = uvw$ such that $\deg(u) = \deg(v) = \deg(w) = 4$.

The proofs of (1) and (2) are clear. If G contains a vertex v of degree at most two and e is an edge incident with v , then $\chi''_{gl}(G - e) \leq k + 2$, by minimality of G , and so there exists an (A, L, f) -coloring c for $T(G - e)$. Erase the color of vertex v in this coloring. There are at least $(k + 2) - (k + 1)$ usable colors in the list of edge e , and so it can be colored. Now since $k \geq 6$, the vertex v can be colored from its list, and this gives an (A, L, f) -coloring for $T(G)$, which contradicts the minimality of G . Also if G contains an edge uv with $\min\{\deg(u), \deg(v)\} \leq \frac{k}{2}$ and $\deg(u) + \deg(v) \leq k + 2$, then any (A, L, f) -coloring of $T(G - uv)$ can be extended to an (A, L, f) -coloring of

$T(G)$, which is contradiction. This shows that (3) and (4) hold. To see (5), on the contrary, let such a face exists. Let $G' = G - \{uv, vw, uw\}$. By the minimality of G , $T(G')$ has an (A, L, f) -coloring c . Erase the colors of u, v, w and, for an element $x \in \{u, v, w, uv, vw, uw\}$, let $L'(x)$ be the available colors in the list of x . Since $k \geq 6$ and since u, v , and w are degree 4 vertices, each $|L'(x)| \geq k - 2 \geq 4$. By Theorem 3.1, it is possible to recolor the elements u, v, w, uv, vw, uw , in this path with colors $c(u) \in L'(u), c(v) \in L'(v), c(w) \in L'(w), c(uv) \in L'(uv), c(vw) \in L'(vw)$ and $c(uw) \in L'(uw)$, respectively, so that c is indeed an (A, L, f) -coloring of $T(G)$, which is a contradiction.

Since G is a planar graph, by Euler's Formula, we have:

$$\sum_{v \in V} (2 \deg(v) - 6) + \sum_{f \in F} (\deg(f) - 6) = -6(|V| - |E| + |F|) = -12.$$

We define the initial charge function $w(x)$ for each $x \in V \cup F$. Let $w(v) = 2 \deg(v) - 6$ if $v \in V$ and $w(f) = \deg(f) - 6$ if $f \in F$. It follows that $\sum_{x \in V \cup F} w(x) < 0$. We construct a new charge $w^*(x)$ on G as follows:

- Each 3-face receives $\frac{3}{2}$ from its incident vertices of degree at least 5.
- Each 3-face receives $\frac{3}{4}$ from its incident vertices of degree 4.
- Each 5-face receives $\frac{1}{3}$ from its incident vertices of degree at least 5.
- Each 5-face receives $\frac{1}{4}$ from its incident vertices of degree 4.

Note that $w^*(f) = w(f) \geq 0$ if $\deg(f) \geq 6$. Assume that $\deg(f) = 3$. If $\delta(f) = 3$, then f is incident with two 6^+ -vertices by (4). So $w^*(f) \geq w(f) + 2 \times \frac{3}{2} = 0$. Otherwise, f is incident with at least one 5^+ -vertex by (5). So $w^*(f) \geq w(f) + \frac{3}{2} + 2 \times \frac{3}{4} = 0$. Let $\deg(f) = 5$. If $\delta(f) = 3$, then f is incident with at most two vertices of degree 3 by (4), and if f is incident with two vertices of degree 3, then f is incident with three 6^+ -vertices. Thus $w^*(f) \geq w(f) + \min\{2 \times \frac{1}{3} + 2 \times \frac{1}{4}, 3 \times \frac{1}{3}\} = 0$. Otherwise, $w^*(f) \geq w(f) + 5 \times \frac{1}{4} > 0$. Let v be a vertex of G . Clearly, $w^*(v) = w(v) = 0$ if $\deg(v) = 3$. If $\deg(v) = 4$, then v is incident with at most two 3-faces by (2). So $w^*(v) \geq w(v) - 2 \times \frac{3}{4} - 2 \times \frac{1}{4} = 0$. If $\deg(v) = 5$, then v is incident with at most two 3-faces by (b). So $w^*(v) \geq w(v) - 2 \times \frac{3}{2} - 3 \times \frac{1}{3} = 0$. If $\deg(v) = 6$, then $w^*(v) \geq w(v) - 3 \times \frac{3}{2} - 3 \times \frac{1}{3} > 0$. If $\deg(v) \geq 7$, so $w^*(v) \geq w(v) - \lfloor \frac{\deg(v)}{2} \rfloor \times \frac{3}{2} - \lceil \frac{\deg(v)}{2} \rceil \times \frac{1}{3} > 0$. It follows that $\sum_{x \in V \cup F} w^*(x) = \sum_{x \in V \cup F} w(x) \geq 0$, a contradiction. ■

A face f is called *simple* if its boundary is a cycle. If $f = u_1 u_2 \dots u_n$ is not simple, then f contains at least one cut vertex v . Let $m_v(f)$ denotes the number of times through v of f in clockwise order. In the sequel, we prove that plane graphs without 4,5-cycles with maximum degree $\Delta \geq 5$ are totally $(\Delta + 2)$ -group choosable.

Theorem 5.7 Let $k \geq 5$ and G be a planar graph with maximum degree $\Delta \leq k$ such that G has no cycles of length 4 and 5. Then $\chi''_{gl}(G) \leq k + 2$.

Proof. Let G be a minimal counterexample. For a $k \geq 5$, a group A with $|A| \geq k + 2$, a $(k + 2)$ -list assignment $L : V(T(G)) \rightarrow 2^A$ and $f \in F(T(G), A)$, $T(G)$ is not (A, L, f) -colorable. Theorem 5.6 implies the theorem when $k \geq 6$. Hence it is sufficient to prove

the theorem when $k = 5$. Graph G has the following properties:

- (1) G is connected,
- (2) Any vertex v is incident with at most $\lfloor \frac{\deg(v)}{2} \rfloor$ 3-faces,
- (3) The minimum degree of G is at least 3 i.e $\delta \geq 3$,
- (4) G contains no edge uv with $\min\{\deg(u), \deg(v)\} = 3$ and $\deg(u) + \deg(v) \leq 7$.

The proofs of (1)-(4) are similar to the proof of Theorem 5.6. We define the initial charge function $w(x) = \deg(x) - 4$ for each $x \in V \cup F$. By the Euler's Formula, we have $\sum_{x \in V \cup F} w(x) = -8$. We construct a new charge $w^*(x)$ on G as follows:

Each $r(\geq 6)$ -face f gives $(1 - \frac{4}{r})m_v(f)$ to its incident vertex v if v is cut vertex, and gives $1 - \frac{4}{r}$ otherwise.

Each 3-vertex v receives $\frac{1}{3}$ from u if v is incident with 3-face f and u is a neighbor of v but not incident with f .

Each 3-face receives $\frac{1}{2}$ from its incident vertex v if $\deg(v) = 5$ and receives $\frac{1}{3}$ if $\deg(v) = 4$.

Note that $w^*(f) \geq 0$ for any face f . Let v be a vertex of G . Suppose that $\deg(v) = 3$. If v is incident with a 3-face f , then v receives at least $\frac{2}{3}$ from its incident faces and $\frac{1}{3}$ from its incident vertex not lying on f . So $w^*(v) \geq w(v) + \frac{2}{3} + \frac{1}{3} = 0$. Otherwise, v receives at least $3 \times \frac{1}{3}$ from its incident faces and hence $w^*(v) \geq w(v) + 1 = 0$. Let $\deg(v) = 4$. The vertex v is incident with at most two 3-faces by (2), so v gives at most $\frac{2}{3}$ to its incident 3-faces. Also v receives at least $\frac{2}{3}$ from its incident faces of degree at least 6. Hence $w^*(v) \geq w(v) + \frac{2}{3} - \frac{2}{3} = 0$. Finally let $\deg(v) = 5$. The vertex v is incident with at most two 3-faces by (2), and if v is incident with exactly two 3-faces and a 3-vertex is pending on the remaining neighbors of v , then v gives at most $2 \times \frac{1}{2} + \frac{1}{3}$ and receives at least $3 \times \frac{1}{3}$ from its faces of degree at least 6. So $w^*(v) \geq w(v) + 3 \times \frac{1}{3} - (2 \times \frac{1}{2} + \frac{1}{3}) > 0$. If v is incident with a 3-face and three 3-vertices is pending on the remaining neighbors of v , then v gives at most $3 \times \frac{1}{3} + \frac{1}{2}$ and receives at least $4 \times \frac{1}{3}$ from its faces of degree at least 6. So $w^*(v) \geq w(v) + 4 \times \frac{1}{3} - (3 \times \frac{1}{3} + \frac{1}{2}) > 0$. Otherwise v gives at most $5 \times \frac{1}{3}$ and receives at least $5 \times \frac{1}{3}$ from its faces of degree at least 6. So $w^*(v) \geq w(v) + \frac{5}{3} - \frac{5}{3} > 0$. It follows that $\sum_{x \in V \cup F} w^*(x) = \sum_{x \in V \cup F} w(x) > 0$, a contradiction. This contradiction completes the proof of the theorem. \blacksquare

Lemma 5.8 ([17]) Let G be a planar graph with $\delta \geq 3$ and no five cycles. Then there exists an edge xy such that $\deg(x) = 3$ and $\deg(y) \leq 5$.

The following theorem proves Conjecture 3 for planar graphs without 5-cycles and maximum degree at least 6.

Theorem 5.9 Let $k \geq 6$ and G be a planar graph with maximum degree $\Delta \leq k$. If G contains no 5-cycles, then $\chi''_{gl}(G) \leq k + 2$.

Proof. Let $G = (V, E)$ be a minimal counterexample for theorem. So for a $k \geq 6$, a group A with $|A| \geq k + 2$, a $(k + 2)$ -list assignment $L : V(T(G)) \rightarrow 2^A$ and

$f \in F(T(G), A)$, $T(G)$ is not (A, L, f) colorable. By Lemma 5.1, we may assume that $\delta \geq 3$. So by Lemma 5.8, G contains an edge xy such that $\deg(x) = 3$ and $\deg(y) \leq 5$. the graph $G - e$ has an (A, L, f) -coloring by the minimality of G . Now erase the color of x in this coloring and color the edge xy from its list, which is possible since its list has at least $(k + 2) - 7 \geq 1$ usable colors. Since $\deg_{T(G)}(x) = 6$ and $k \geq 6$, the vertex x can be colored easily. So G has an (A, L, f) -coloring, which is a contradiction. ■

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